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The triangle intersection numbers of a pair of disjoint $S(2, 4, v)$s[☆]

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ABSTRACT

In this paper the triangle intersection problem for a pair of disjoint $S(2, 4, v)$s is investigated. Let $J_T^*(v)$ denote the set of all integers s such that there exists a pair of disjoint $S(2, 4, v)$s intersecting in s triangles. Let $b_v = v(v-1)/12$. We establish that for any positive integer $v \equiv 1, 4 \pmod{12}$, $v \geq 16$ and $v \neq 25, 28, 37$, $J_T^*(v) = [0, b_v]$.

1. Introduction

Let H be a simple graph and G a subgraph of H. A *G-design* of H (or (H, G)-design) is a pair $(X, \mathcal{B})$ where X is the vertex set of H and $\mathcal{B}$ is an edge-disjoint decomposition of H into isomorphic copies (called *blocks*) of the graph G. If H is the complete graph K_v, we refer to such a G-design as one of order v. If G is the complete graph K_k, a K_k-design of order v is called a *Steiner system* $S(2, k, v)$. It is well known that an $S(2, 4, v)$ exists if and only if $v \equiv 1, 4 \pmod{12}$ (cf. [10]).

Two G-designs of order v, $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$, are said to *intersect* in s blocks provided $|\mathcal{B}_1 \cap \mathcal{B}_2| = s$. If $s = 0$, $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ are said to be *disjoint*. The *intersection problem* for G-designs is the determination of all integral pairs (v, s) such that there exists a pair of G-designs of order v intersecting in s blocks. This problem was first introduced for $S(2, k, v)$s (cf. [11]). A complete solution to the intersection problem for $S(2, 3, v)$s was made by Lindner and Rosa [14]. The intersection problem for $S(2, 4, v)$s was dealt with by Colbourn et al. [8], apart from three undecided values for $v = 25$, 28 and 37. The intersection problem is also considered for many other different types of combinatorial structures. The interested reader may refer to [3,9], for example.

Let B be a simple graph. Denote by $T(B)$ the set of all triangles of the graph B. For example, if B is the graph with vertices a, b, c, d and edges ab, ac, bc, cd (such a graph called a *kite*), then $T(B) = \{\{a, b, c\}\}$. Two G-designs of order $v(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ are said to *intersect* in s triangles provided $|T(\mathcal{B}_1) \cap T(\mathcal{B}_2)| = s$, where $T(\mathcal{B}_i) = \bigcup_{B \in \mathcal{B}_i} T(B)$, $i = 1, 2$. The *triangle intersection problem* for G-designs is the determination of all integral pairs (v, s) such that there exists a pair of G-designs of order v intersecting in s triangles.

The triangle intersection problem was considered by Lindner and Yazici [13], who gave a complete solution to the triangle intersection problem for kite systems (a kite system is a G-design when G is a kite). Billington et al. [4] discussed the triangle intersection problem for $(K_4 - e)$-designs. The authors [7] investigated the triangle intersection problem for $S(2, 4, v)$s. We record the result as follows.

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Theorem 1.1 ([7]). *Let $J_T(v) = \{s$: there exists a pair of $S(2, 4, v)$ s intersecting in s triangles$\}$. Let $I_T(v) = \{0, 1, \ldots, t_v - 30\} \cup \{t_v - 27, t_v - 24, t_v - 18, t_v\}$ where $t_v = v(v-1)/3$. Then*

(1) *For $v \equiv 1, 4 \pmod{12}$ and $v \geq 121$, $J_T(v) = I_T(v)$; In particular, $J_T(40) = I_T(40)$.*
(2) *For $v \equiv 1, 4 \pmod{12}$ and $49 \leq v \leq 112$, $I_T(v) \setminus \{t_v - 33\} \subseteq J_T(v) \subseteq I_T(v)$.*
(3) *$J_T(13) = I_T(13) \setminus \{1, 2, 19\}$ and $J_T(16) = I_T(16) \setminus \{37, 39, 41, 43, 45\text{–}50, 53, 62\}$.*
(4) *$\{0\text{–}122, 124\text{–}131, 134, 135, 137, 140, 143, 146, 155, 158, 164, 200\} \subseteq J_T(25) \subseteq I_T(25) \setminus \{176, 182\}$.*
(5) *$\{0\text{–}149, 156, 158, 160, 162, 164, 166, 168, 180, 204, 252\} \subseteq J_T(28) \subseteq I_T(28)$.*
(6) *$\{0\text{–}251, 258\text{–}276, 285\text{–}294, 444\} \subseteq J_T(37) \subseteq I_T(37)$.*

If two $S(2, 4, v)$s have blocks in common, each common block contributes 4 common triangles. It is natural to ask how about the triangle intersection problem for a pair of disjoint $S(2, 4, v)$s. In this paper, we consider all possible triangle intersection numbers of a pair of disjoint $S(2, 4, v)$s. In what follows we always assume that $b_v = v(v-1)/12$ (the number of blocks in an $S(2, 4, v)$) and $[a, b]$ denote the set of integers x such that $a \leq x \leq b$. Let $J_T^*(v) = \{s$: there exists a pair of disjoint $S(2, 4, v)$s intersecting in s triangles$\}$. It is easy to see that $J_T^*(v) \subseteq [0, b_v]$. As the main result of the present paper, we are to show the following theorem.

Theorem 1.2. (1) *For $v \equiv 1, 4 \pmod{12}$, $v \geq 16$ and $v \neq 25, 28, 37$, $J_T^*(v) = [0, b_v]$.*
(2) $J_T^*(13) = [0, 12] \setminus \{1, 2\}$.
(3) $[0, 43] \cup \{50\} \subseteq J_T^*(25)$.
(4) $[0, 56] \cup \{63\} \subseteq J_T^*(28)$.
(5) $[0, 106] \cup \{111\} \subseteq J_T^*(37)$.

2. Recursive constructions

In this section we present two recursive constructions for the triangle intersection problem for a pair of disjoint G-designs. The concept of GDDs plays an important role in these constructions.

Let K be a set of positive integers. A *group divisible design* (GDD) K-GDD is a triple $(X, \mathcal{G}, \mathcal{A})$ satisfying the following properties: (1) $\mathcal{G}$ is a partition of a finite set X into subsets (called *groups*); (2) $\mathcal{A}$ is a set of subsets of X (called *blocks*), each of cardinality from K, such that a group and a block contain at most one common point; (3) every pair of points from distinct groups occurs in exactly one block.

If $\mathcal{G}$ contains u_i groups of size g_i for $1 \leq i \leq s$, then we call $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ the *group type* (or *type*) of the GDD. If $K = \{k\}$, we write $\{k\}$-GDD as k-GDD. A K-GDD of type 1^v is commonly called a *pairwise balanced design*, denoted by $(v, K, 1)$-PBD. When $K = \{k\}$, a pairwise balanced design is just a Steiner system $S(2, k, v)$, called a *balanced incomplete block design*, denoted by $(v, k, 1)$-BIBD. A K-GDD of type $1^{v-h}h^1$ is commonly called an *incomplete pairwise balanced design*, denoted by $(v, h; K, 1)$-IPBD. When $K = \{k\}$, an incomplete pairwise balanced design is called an *incomplete balanced incomplete block design*, denoted by $(v, h; k, 1)$-IBIBD. Obviously a $(v, h; k, 1)$-IBIBD is also a $((K_v \setminus K_h), K_k)$-design.

A GDD is *resolvable* if its blocks can be partitioned into parallel classes; a parallel class is a set of point-disjoint blocks whose union is the set of all points. The notation K-RGDD is used for a resolvable K-GDD. If $K = \{k\}$, we write $\{k\}$-RGDD as k-RGDD. A 3-RGDD of type 1^v is commonly called a *Kirkman triple system*, denoted by KTS(v). It is well known that a KTS(v) exists if and only if $v \equiv 3 \pmod 6$ [16].

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ be a partition of a finite set X into subsets (called *holes*), where $|H_i| = n_i$ for $1 \leq i \leq t$. Let $K_{n_1,n_2,\ldots,n_t}$ be the complete multipartite graph on X with the i-th part on H_i. A *holely G-design* is a triple $(X, \mathcal{H}, \mathcal{B})$ such that $(X, \mathcal{B})$ is a $(K_{n_1,n_2,\ldots,n_t}, G)$-design. The *hole type* (or *type*) of the holely G-design is $\{n_1, n_2, \ldots, n_t\}$. We usually use an "exponential" notation to describe hole types: the hole type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ denotes u_i occurrences of g_i for $1 \leq i \leq s$.

A pair of holely G-designs $(X, \mathcal{H}, \mathcal{B}_1)$ and $(X, \mathcal{H}, \mathcal{B}_2)$ of the same type are said to *disjoint* if $|\mathcal{B}_1 \cap \mathcal{B}_2| = 0$. A pair of holely G-designs $(X, \mathcal{H}, \mathcal{B}_1)$ and $(X, \mathcal{H}, \mathcal{B}_2)$ are said to *intersect in l triangles* if $|T(\mathcal{B}_1) \cap T(\mathcal{B}_2)| = l$, where $T(\mathcal{B}_i) = \bigcup_{B \in \mathcal{B}_i} T(B)$, $i = 1, 2$. The following construction is a variation of Wilson's Fundamental Construction [17].

Construction 2.1 (*Weighting Construction*). Suppose that $(X, \mathcal{G}, \mathcal{A})$ is a K-GDD, and let $\omega : X \longmapsto Z^+ \cup \{0\}$ be a weight function. For every block $A \in \mathcal{A}$, suppose that there is a pair of disjoint holely G-designs of type $\{\omega(x) : x \in A\}$, which intersect in t_A triangles. Then there exists a pair of disjoint holely G-designs of type $\{\sum_{x \in G} \omega(x) : G \in \mathcal{G}\}$, which intersect in $\sum_{A \in \mathcal{A}} t_A$ triangles.

Proof. For every $x \in X$, let $S(x)$ be a set of $\omega(x)$ "copies" of x. For any $Y \subseteq X$, let $S(Y) = \bigcup_{x \in Y} S(x)$. For every block $A \in \mathcal{A}$, construct a pair of disjoint holely G-designs $\{S(A), \{S(x) : x \in A\}, \mathcal{B}_A\}$ and $\{S(A), \{S(x) : x \in A\}, \mathcal{B}'_A\}$, which intersect in t_A triangles. Then it is readily checked that there exists a pair of disjoint holely G-designs $(S(X), \{S(G) : G \in \mathcal{G}\}, \cup_{A \in \mathcal{A}} \mathcal{B}_A)$ and $(S(X), \{S(G) : G \in \mathcal{G}\}, \cup_{A \in \mathcal{A}} \mathcal{B}'_A)$, which intersect in $\sum_{A \in \mathcal{A}} t_A$ triangles. □

Construction 2.2 (*Filling Construction*). Let a be a nonnegative integer. Suppose that there exists a pair of disjoint holely G-designs of type $\{g_1, g_2, \ldots, g_s\}$, which intersect in t triangles. If there is a pair of disjoint $((K_{g_i+a} \setminus K_a), G)$-designs with the same subgraph K_a removed for each $1 \leq i \leq s-1$, which intersect in t_i triangles, and there is a pair of disjoint (K_{g_s+a}, G)-designs, which intersect in t_s triangles, then there exists a pair of disjoint (K_{v+a}, G)-designs intersecting in $t + \sum_{i=1}^{s} t_i$ triangles, where $v = \sum_{i=1}^{s} g_i$.

Proof. Let $(X, \mathcal{H}, \mathcal{B}_1)$ and $(X, \mathcal{H}, \mathcal{B}_2)$ be a pair of disjoint holely G-designs of type $\{g_1, g_2, \ldots, g_s\}$, which intersect in t triangles. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_s\}$ with $|H_i| = g_i$ for $1 \le i \le s$, and Y be a set of cardinality a such that $X \cap Y = \emptyset$. For each $1 \le i \le s-1$, construct a pair of disjoint $((K_{g_i+a} \setminus K_a), G)$-designs $(H_i \bigcup Y, \mathcal{A}_i^1)$ and $(H_i \bigcup Y, \mathcal{A}_i^2)$ with the same subgraph K_a defined on Y removed, which intersect in t_i triangles. By the assumption, we also have a pair of disjoint (K_{g_s+a}, G)-designs $(H_s \bigcup Y, \mathcal{A}_s^1)$ and $(H_s \bigcup Y, \mathcal{A}_s^2)$, which intersect in t_s triangles. It is readily checked that there exists a pair of disjoint (K_{v+a}, G)-designs $(X \bigcup Y, (\cup_{i=1}^s \mathcal{A}_i^1) \bigcup \mathcal{B}_1)$ and $(X \bigcup Y, (\cup_{i=1}^s \mathcal{A}_i^2) \bigcup \mathcal{B}_2)$, which intersect in $t + \sum_{i=1}^s t_i$ triangles. □

It is well known that a 5-GDD of type g^5 is equivalent to three mutually orthogonal Latin squares (MOLS) of order g. Thus we quote the following result for later use.

Lemma 2.3 ([1]). *There exists a 5-GDD of type g^5 for any positive integer $g \ge 4$ and $g \ne 6, 10$.*

Lemma 2.4 ([2]). *For any positive integer $v \equiv 0, 1 \pmod 4$ and $v \ne 8, 9, 12$, there exists a $(v, \{4, 5\}, 1)$-PBD.*

Lemma 2.5 ([6]). *The necessary and sufficient conditions for the existence of a 4-GDD of type g^n are (1) $n \ge 4$, (2) $(n-1)g \equiv 0 \pmod 3$, (3) $n(n-1)g^2 \equiv 0 \pmod{12}$, with the exception of $(g, n) \in \{(2, 4), (6, 4)\}$, in which case no such GDD exists.*

3. Working lemmas

If two $S(2, 4, v)$s have one block in common, then they intersect in at least 4 triangles. By Theorem 1.1, we have the following lemma.

Lemma 3.1. (1) $0, 1, 2, 3 \in J_T^*(v)$ *for any* $v \equiv 1, 4 \pmod{12}$ *and* $v \ge 16$;
(2) $0, 3 \in J_T^*(13)$, $1, 2 \notin J_T^*(13)$.

Lemma 3.2. *Let $J_1^*(13) = \{s :$ there exist two $S(2, 4, 13)$s with $s + 4$ common triangles and exactly one common block$\}$. Then $J_1^*(13) = [4, 12]$.*

Proof. Let $X = Z_{13}$ and $\mathcal{B} = \{\{i, 1+i, 3+i, 9+i\} : 0 \le i \le 12\}$. Then $(X, \mathcal{B})$ is an $S(2, 4, 13)$. Consider the following permutations on X.

$$\pi_4 = (3\,4)(6\,8)(7\,10)(9\,11), \qquad \pi_5 = (3\,4)(6\,8)(7\,10)(9\,11\,12),$$
$$\pi_6 = (3\,4)(6\,8)(7\,10\,9\,11), \qquad \pi_7 = (3\,4)(6\,9\,10\,12\,11\,8\,7),$$
$$\pi_8 = (3\,4)(6\,9\,7)(11\,12), \qquad \pi_9 = (3\,4)(6\,9\,10\,12\,11\,7),$$
$$\pi_{10} = (3\,4)(6\,9\,10\,8), \qquad \pi_{11} = (3\,4)(6\,9\,10\,11),$$
$$\pi_{12} = (3\,4)(6\,12).$$

It is readily checked that $|\pi_k\mathcal{B} \bigcap \mathcal{B}| = 1$ and $|T(\pi_k\mathcal{B} \setminus \mathcal{B}) \bigcap T(\mathcal{B} \setminus \pi_k\mathcal{B})| = k$ for each $k \in [4, 12]$. Note that there is a unique $S(2, 4, 13)$ up to isomorphism. With computer exhaustive search we have that $0, 1, 2, 3 \notin J_1^*(13)$. □

Lemma 3.3. $J_T^*(13) = [0, 12] \setminus \{1, 2\}$.

Proof. Take the $S(2, 4, 13)$ $(X, \mathcal{B})$ given in Lemma 3.2. Consider the following permutations on X.

$$\pi_4 = (3\,4)(6\,7\,10\,12\,11\,9\,8), \qquad \pi_5 = (3\,4)(6\,7\,10\,9\,11\,8),$$
$$\pi_6 = (3\,4)(6\,7\,10\,9\,11\,12\,8), \qquad \pi_7 = (3\,4)(6\,7\,10\,9\,12\,8),$$
$$\pi_8 = (3\,4)(6\,7\,10\,11\,12\,9), \qquad \pi_9 = (3\,4)(6\,7\,10\,12)(8\,9),$$
$$\pi_{10} = (3\,4)(6\,7\,11\,10\,9\,8), \qquad \pi_{11} = (3\,4)(6\,7\,11\,10\,8),$$
$$\pi_{12} = (3\,4)(6\,7\,11\,10\,8\,12).$$

It is readily checked that $(X, \pi_k\mathcal{B})$ and $(X, \mathcal{B})$ are disjoint and $|T(\pi_k\mathcal{B}) \bigcap T(\mathcal{B})| = k$ for each $k \in [4, 12]$. With computer exhaustive search we have that $13 \notin J_T^*(13)$. Combining the result from Lemma 3.1, we complete the proof. □

Lemma 3.4. *Let $J_1^*(16) = \{s :$ there exist two $S(2, 4, 16)$s with $s + 4$ common triangles and exactly one common block$\}$. Then $J_1^*(16) = [0, 19]$.*

Proof. Construct an $S(2, 4, 16)$ $(X, \mathcal{B})$ with $X = Z_{16}$. All blocks of $\mathcal{B}$ are listed below, which can be found in Example 1.31 in [15].

{0, 1, 2, 3},	{0, 4, 5, 6},	{0, 7, 8, 9},	{0, 10, 11, 12},	{0, 13, 14, 15},
{1, 4, 7, 10},	{1, 5, 11, 13},	{1, 6, 8, 14},	{1, 9, 12, 15},	{2, 4, 12, 14},
{2, 5, 7, 15},	{2, 6, 9, 11},	{2, 8, 10, 13},	{3, 4, 9, 13},	{3, 5, 8, 12},
{3, 6, 10, 15},	{3, 7, 11, 14},	{4, 8, 11, 15},	{5, 9, 10, 14},	{6, 7, 12, 13}.

Consider the following permutations on X.

$\pi_0 = (0\ 13\ 8\ 2\ 14\ 4\ 5\ 15\ 1\ 10)(3\ 12\ 11\ 9)$, $\pi_1 = (0\ 13\ 8)(1\ 10\ 4\ 5\ 15\ 9\ 3\ 12\ 7\ 11\ 2\ 14)$,
$\pi_2 = (0\ 13\ 9\ 4\ 5\ 15\ 1\ 10\ 3\ 12\ 2\ 14\ 8)(7\ 11)$, $\pi_3 = (0\ 13\ 9\ 4\ 5\ 15\ 1\ 10\ 3\ 12\ 7\ 11\ 8)(2\ 14)$,
$\pi_4 = (0\ 13\ 9\ 7\ 11\ 4\ 5\ 15\ 1\ 10\ 3\ 12\ 2\ 14\ 8)$, $\pi_5 = (0\ 13\ 9\ 7\ 11\ 8)(1\ 10\ 3\ 12\ 4\ 5\ 15)(2\ 14)$,
$\pi_6 = (0\ 13\ 9\ 7\ 11\ 1\ 10\ 4\ 5\ 15\ 8)(2\ 14\ 3\ 12)$, $\pi_7 = (0\ 13\ 9\ 7\ 11\ 3\ 12\ 2\ 14\ 1\ 10\ 4\ 5\ 15\ 8)$,
$\pi_8 = (0\ 13\ 2\ 14\ 1\ 10\ 4\ 5\ 15\ 8)(3\ 12)(7\ 11\ 9)$, $\pi_9 = (0\ 13\ 2\ 14\ 3\ 12\ 9\ 7\ 11\ 1\ 10\ 8)(4\ 5\ 15)$,
$\pi_{10} = (0\ 13\ 2\ 14\ 4\ 5\ 15\ 3\ 12\ 9\ 7\ 11\ 1\ 10\ 8)$, $\pi_{11} = (0\ 13\ 9\ 7\ 11\ 2\ 14\ 4\ 5\ 15\ 3\ 12\ 1\ 10\ 8)$,
$\pi_{12} = (0\ 13\ 8)(1\ 10\ 9\ 7\ 11\ 3\ 12)(2\ 14)(4\ 5\ 15)$, $\pi_{13} = (0\ 13\ 2\ 14\ 4\ 5\ 15\ 7\ 11\ 3\ 12\ 9\ 8)(1\ 10)$,
$\pi_{14} = (0\ 13\ 3\ 12\ 9)(1\ 10\ 2\ 14\ 8)(4\ 5\ 15\ 7\ 11)$, $\pi_{15} = (0\ 13\ 4\ 5\ 15\ 7\ 11\ 8\ 1\ 10\ 2\ 14\ 3\ 12\ 9)$,
$\pi_{16} = (0\ 13\ 4\ 5\ 15\ 7\ 11\ 8\ 2\ 14\ 3\ 12\ 9\ 1\ 10)$, $\pi_{17} = (0\ 13\ 4\ 5\ 15\ 11\ 3\ 12\ 9\ 2\ 14\ 8\ 1\ 10\ 6\ 7)$,
$\pi_{18} = (0\ 13\ 4\ 5\ 15\ 8\ 1\ 10\ 6\ 7)(2\ 14\ 11\ 3\ 12\ 9)$, $\pi_{19} = (0\ 6\ 12\ 8\ 2\ 15\ 1\ 4\ 11\ 5\ 9\ 7\ 13\ 10\ 3)$.

It is readily checked that $|\pi_k\mathcal{B} \bigcap \mathcal{B}| = 1$ and $|T(\pi_k\mathcal{B} \setminus \mathcal{B}) \bigcap T(\mathcal{B} \setminus \pi_k\mathcal{B})| = k$ for each $k \in [0, 19]$. □

Lemma 3.5. $J_T^*(16) = [0, 20]$.

Proof. Take the $S(2, 4, 16)$ $(X, \mathcal{B})$ constructed in Lemma 3.4. Consider the following permutations on X.

$\pi_4 = (0\ 12\ 11\ 4\ 5\ 14\ 13\ 10\ 7\ 8\ 2\ 3)(1\ 15\ 9)$, $\pi_5 = (0\ 12\ 11\ 4\ 5\ 14\ 9\ 1\ 15\ 10\ 7\ 8\ 2\ 3)$,
$\pi_6 = (0\ 12\ 10\ 7\ 8\ 2\ 3)(1\ 15\ 13\ 4\ 5\ 14\ 11\ 9)$, $\pi_7 = (0\ 12\ 9\ 1\ 15\ 11\ 10\ 7\ 8\ 2\ 3)(4\ 5\ 14\ 13)$,
$\pi_8 = (0\ 12\ 9\ 1\ 15\ 11\ 10\ 7\ 8\ 2\ 3)(4\ 5\ 14)$, $\pi_9 = (0\ 12\ 9\ 1\ 15\ 4\ 5\ 14\ 11\ 10\ 7\ 8\ 2\ 3)$,
$\pi_{10} = (0\ 12\ 11\ 10\ 7\ 8\ 2\ 3)(1\ 15\ 13\ 9)(4\ 5\ 14)$, $\pi_{11} = (0\ 12\ 13\ 11\ 10\ 7\ 8\ 2\ 3)(1\ 15\ 9)(4\ 5\ 14)$,
$\pi_{12} = (0\ 12\ 9\ 1\ 15\ 11\ 13\ 4\ 5\ 14\ 10\ 7\ 8\ 2\ 3)$, $\pi_{13} = (0\ 12\ 9\ 1\ 15\ 4\ 5\ 14\ 10\ 7\ 8\ 2\ 3)(11\ 13)$,
$\pi_{14} = (0\ 12\ 4\ 5\ 14\ 10\ 9\ 1\ 15\ 7\ 8\ 2\ 3)(11\ 13)$, $\pi_{15} = (0\ 12\ 9\ 4\ 5\ 14\ 10\ 1\ 15\ 11\ 13\ 7\ 8\ 2\ 3)$,
$\pi_{16} = (0\ 12\ 13\ 1\ 15\ 11\ 9\ 4\ 5\ 14\ 10\ 7\ 8\ 2\ 3)$, $\pi_{17} = (0\ 12\ 4\ 5\ 14\ 9\ 10\ 7\ 8\ 2\ 3)(1\ 15)(11\ 13)$,
$\pi_{18} = (0\ 12\ 1\ 15\ 8\ 13\ 4\ 5\ 14\ 2\ 3)(6\ 7)$, $\pi_{19} = (0\ 12\ 13\ 6\ 8\ 9\ 2\ 3)(1\ 15\ 7)(4\ 5\ 14)(10\ 11)$,
$\pi_{20} = (0\ 10\ 9\ 15\ 3\ 4\ 5\ 12\ 7\ 8\ 13\ 6)(1\ 14\ 11)$.

It is readily checked that $(X, \pi_k\mathcal{B})$ and $(X, \mathcal{B})$ are disjoint and $|T(\pi_k\mathcal{B}) \bigcap T(\mathcal{B})| = k$ for each $k \in [4, 20]$. Combining the result from Lemma 3.1, we complete the proof. □

Lemma 3.6. $50 \in J_T^*(25)$.

Proof. Two $S(2, 4, 25)$s $(Z_5 \times Z_5, \mathcal{B}_i)$, $i = 1, 2$, are constructed by listing their base blocks below. All other blocks are obtained by developing these base blocks in $Z_5 \times Z_5$.

$\mathcal{B}_1$: $\{(0, 0), (0, 1), (1, 0), (4, 4)\}, \{(0, 0), (0, 2), (2, 0), (3, 3)\}$.
$\mathcal{B}_2$: $\{(0, 0), (0, 1), (1, 0), (2, 2)\}, \{(0, 0), (0, 2), (2, 0), (4, 4)\}$.

It is readily checked $\mathcal{B}_1 \bigcap \mathcal{B}_2 = \emptyset$ and $|T(\mathcal{B}_1) \bigcap T(\mathcal{B}_2)| = 50$. □

Lemma 3.7. $63 \in J_T^*(28)$.

Proof. Two $S(2, 4, 28)$s $(Z_{28}, \mathcal{B}_i)$, $i = 1, 2$, are constructed by listing their base blocks below. All other blocks are obtained by developing these base blocks by $+4$ modulo 28.

$\mathcal{B}_1$:	$\{0, 1, 2, 3\}$,	$\{0, 4, 9, 12\}$,	$\{0, 6, 11, 14\}$,	$\{0, 7, 18, 21\}$,
	$\{0, 10, 17, 23\}$,	$\{0, 13, 22, 26\}$,	$\{0, 15, 19, 27\}$,	$\{1, 5, 13, 23\}$,
	$\{1, 6, 18, 27\}$.			
$\mathcal{B}_2$:	$\{1, 2, 3, 14\}$,	$\{0, 1, 4, 9\}$,	$\{2, 4, 7, 24\}$,	$\{2, 5, 12, 26\}$,
	$\{0, 2, 10, 23\}$,	$\{1, 5, 10, 16\}$,	$\{0, 7, 15, 16\}$,	$\{1, 7, 11, 17\}$,
	$\{1, 15, 18, 27\}$.			

It is readily checked $\mathcal{B}_1 \bigcap \mathcal{B}_2 = \emptyset$ and $|T(\mathcal{B}_1) \bigcap T(\mathcal{B}_2)| = 63$. □

Lemma 3.8. *There exists a pair of disjoint 4-GDDs of type 3^5 with i common triangles, where $i \in \{0, 15\}$.*

Proof. The case of $i = 0$ follows from Lemma 4.10 in [7]. We next deal with the case of $i = 15$. Take the $S(2, 4, 16)$ $(X, \mathcal{B})$ constructed in Lemma 3.4. Delete the point 0 from this design to obtain a 4-GDD of type 3^5 $(X \setminus \{0\}, \mathcal{G}, \mathcal{B}')$, where $\mathcal{G} = \{\{1 + 3j, 2 + 3j, 3 + 3j\} : 0 \le j \le 4\}$, and $\mathcal{B}' = \mathcal{B} \setminus \{B \in \mathcal{B} : 0 \in B\}$. Consider the permutation $\pi = (5\ 6)(8\ 9)(10\ 11\ 12)(13\ 15)$ on $X \setminus \{0\}$, which keeps $\mathcal{G}$ invariant. It is readily checked that $\pi\mathcal{B}'$ and $\mathcal{B}'$ are disjoint, and $|T(\pi\mathcal{B}') \bigcap T(\mathcal{B}')| = 15$. □

Lemma 3.9. (1) *There exists a pair of disjoint 4-GDDs of type g^4 without common triangles for $g \in \{4, 5, 9\}$;*
(2) *There exists a pair of disjoint 4-GDDs of type g^4 with g^2 common triangles for $g \in \{3, 4, 5, 9\}$.*

Proof. The first assertion follows immediately from Lemma 4.11 in [7]. We prove the second assertion as follows. For $g \in \{3, 4, 5, 9\}$, by Lemma 2.5 there exists a 4-GDD of type g^4, which is equivalent to the existence of a 3-RGDD of type g^3. Let $(I_g \times \{0, 1, 2\}, \{I_g \times \{i\} : i = 0, 1, 2\}, \mathcal{A})$ be the 3-RGDD of type g^3 and $\mathcal{A}$ can be partitioned into g parallel classes $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_g$. Let $X = (I_g \times \{0, 1, 2\}) \cup \{\infty_1, \infty_2, \ldots, \infty_g\}$ and $\mathcal{G} = \{I_g \times \{i\} : i = 0, 1, 2\} \cup \{\{\infty_1, \infty_2, \ldots, \infty_g\}\}$. Define two block sets

$$\mathcal{B}_1 = \bigcup_{j=1}^{g} \{\{\infty_j\} \cup T : T \in \mathcal{P}_j\}$$

and

$$\mathcal{B}_2 = \bigcup_{j=1}^{g} \{\{\infty_{j+1}\} \cup T : T \in \mathcal{P}_j\},$$

where the subscript of ∞_{j+1} is taken modulo g. Then $(X, \mathcal{G}, \mathcal{B}_1)$ and $(X, \mathcal{G}, \mathcal{B}_2)$ are 4-GDDs of type g^4. It is readily checked that they are disjoint and $|T(\mathcal{B}_1) \bigcap T(\mathcal{B}_2)| = g^2$. $\square$

4. Applying the recursions

Lemma 4.1. *For any positive integer $v \equiv 1, 13 \pmod{48}$ and $v \geq 49$, $[0, b_v - (v-1)/12] \subseteq J_T^*(v)$.*

Proof. Let $v = 12n + 1$ with $n \equiv 0, 1 \pmod 4$ and $n \geq 4$. Start from a 4-GDD of type 3^n from Lemma 2.5, which contains exactly $x = 3n(n-1)/4$ blocks of size 4. Give each point of the GDD weight 4. By Lemma 3.9, there is a pair of disjoint 4-GDDs of type 4^4 with α common triangles, $\alpha \in \{0, 16\}$. Then apply Construction 2.1 to obtain a pair of disjoint 4-GDDs of type 12^n with $\sum_{i=1}^{x} \alpha_i$ common triangles, where $\alpha_i \in \{0, 16\}$ for $1 \leq i \leq x$. By Construction 2.2, filling in the holes by a pair of disjoint $S(2, 4, 13)$s with β_j $(1 \leq j \leq n)$ common triangles from Lemma 3.3, we have a pair of disjoint $S(2, 4, 12n + 1)$s with $\sum_{i=1}^{x} \alpha_i + \sum_{j=1}^{n} \beta_j$ common triangles, where $\beta_j \in J_T^*(13)$ for $1 \leq j \leq n$. It is readily checked that for any integer $a \in [4, b_v - (v-1)/12]$, a can be written as the form of $\sum_{i=1}^{x} \alpha_i + \sum_{j=1}^{n} \beta_j$, where $\alpha_i \in \{0, 16\}$ $(1 \leq i \leq x)$, $\beta_j \in J_T^*(13)$ $(1 \leq j \leq n)$. Combining the result from Lemma 3.1, we complete the proof. $\square$

Lemma 4.2 ([5]). *There exists a $(v, \{4, 7^*\}, 1)$-PBD with exactly one block of size 7 for any positive integer $v \equiv 7, 10 \pmod{12}$ and $v \neq 10, 19$.*

Lemma 4.3. *For any positive integer $v \equiv 25, 37 \pmod{48}$ and $v \geq 73$, $[0, b_v - (v-1)/12] \subseteq J_T^*(v)$.*

Proof. Let $v = 12n + 1$ with $n \equiv 2, 3 \pmod 4$ and $n \geq 7$. There exists a $(3n + 1, \{4, 7^*\}, 1)$-PBD with exactly one block of size 7 from Lemma 4.2. Take a point from the block of size 7. Delete this point to obtain a 4-GDD of type $3^{n-2}6^1$, which contains $x = 3(n^2 - n - 2)/4$ blocks of size 4. Give each point of the GDD weight 4. By Lemma 3.9, there is a pair of disjoint 4-GDDs of type 4^4 with α common triangles, $\alpha \in \{0, 16\}$. Then apply Construction 2.1 to obtain a pair of disjoint 4-GDDs of type $12^{n-2}24^1$ with $\sum_{i=1}^{x} \alpha_i$ common triangles, where $\alpha_i \in \{0, 16\}$ for $1 \leq i \leq x$. By Construction 2.2, filling in the holes by a pair of disjoint $S(2, 4, 13)$s with β_j $(1 \leq j \leq n-2)$ common triangles from Lemma 3.3, and a pair of disjoint $S(2, 4, 25)$s with β_{n-1} common triangles from Lemmas 3.1 and 3.6, we have a pair of disjoint $S(2, 4, 12n+1)$s with $\sum_{i=1}^{x} \alpha_i + \sum_{j=1}^{n-2} \beta_j + \beta_{n-1}$ common triangles, where $\beta_j \in J_T^*(13)$ for $1 \leq j \leq n-2$ and $\beta_{n-1} \in \{0, 1, 2, 3, 50\} \subseteq J_T^*(25)$. It is readily checked that for any integer $a \in [0, b_v - (v-1)/12]$, a can be written as the form of $\sum_{i=1}^{x} \alpha_i + \sum_{j=1}^{n-2} \beta_j + \beta_{n-1}$, where $\alpha_i \in \{0, 16\}$ $(1 \leq i \leq x)$, $\beta_j \in J_T^*(13)$ $(1 \leq j \leq n-2)$, $\beta_{n-1} \in \{0, 1, 2, 3, 50\}$.

When $v = 73$, start from an $S(2, 5, 25)$ (which comes from a 5-GDD of type 5^5 by Lemma 2.3). Delete a point from this design to obtain a 5-GDD of type 4^6. Give each point of the GDD weight 3. By Lemma 3.8, there is a pair of disjoint 4-GDDs of type 3^5 with α common triangles, $\alpha \in \{0, 15\}$. Then apply Construction 2.1 to obtain a pair of disjoint 4-GDDs of type 12^6 with $\sum_{i=1}^{24} \alpha_i$ common triangles, where $\alpha_i \in \{0, 15\}$ for $1 \leq i \leq 24$. By Construction 2.2, filling in the holes by a pair of disjoint $S(2, 4, 13)$s with β_j $(1 \leq j \leq 6)$ common triangles from Lemma 3.3, we have a pair of disjoint $S(2, 4, 73)$s with $\sum_{i=1}^{24} \alpha_i + \sum_{j=1}^{6} \beta_j$ common triangles, where $\beta_j \in J_T^*(13)$ for $1 \leq j \leq 6$. It is readily checked that for any integer $a \in [4, b_v - (v-1)/12]$, a can be written as the form of $\sum_{i=1}^{24} \alpha_i + \sum_{j=1}^{6} \beta_j$, where $\alpha_i \in \{0, 15\}$ $(1 \leq i \leq 24)$, $\beta_j \in J_T^*(13)$ $(1 \leq j \leq 6)$. Combining the result from Lemma 3.1 we have $[0, b_v - (v-1)/12] \subseteq J_T^*(v)$. $\square$

Lemma 4.4. *For any positive integer $v \equiv 4 \pmod{12}$ and $v \geq 52$, $J_T^*(v) = [0, b_v]$.*

Proof. When $v \equiv 4, 16 \pmod{48}$ and $v \geq 52$, let $v = 12n + 4$ with $n \equiv 0, 1 \pmod 4$ and $n \geq 4$. By similar arguments as in Lemma 4.1, there is a pair of disjoint 4-GDDs of type 12^n with $\sum_{i=1}^{x} \alpha_i$ common triangles, where $x = 3n(n-1)/4$ and $\alpha_i \in \{0, 16\}$ for $1 \leq i \leq x$. By Construction 2.2, filling in the holes by a pair of $S(2, 4, 16)$s with $\beta_j + 4$ $(1 \leq j \leq n-1)$ common triangles and exactly one common block from Lemma 3.4, and a pair of disjoint $S(2, 4, 16)$s with β_n common triangles from

Lemma 3.5, we have a pair of disjoint $S(2, 4, 12n+4)$s with $\sum_{i=1}^{x}\alpha_i + \sum_{j=1}^{n-1}\beta_j + \beta_n$ common triangles, where $\beta_j \in J_1^*(16)$ for $1 \le j \le n-1$ and $\beta_n \in J_T^*(16)$. It is readily checked that for any integer $a \in [0, b_v]$, a can be written as the form of $\sum_{i=1}^{x}\alpha_i + \sum_{j=1}^{n-1}\beta_j + \beta_n$, where $\alpha_i \in \{0, 16\}$ $(1 \le i \le x)$, $\beta_j \in J_1^*(16) = [0, 19]$ $(1 \le j \le n-1)$, $\beta_n \in J_T^*(16) = [0, 20]$.

When $v \equiv 28, 40 \pmod{48}$ and $v \ge 88$, let $v = 12n+4$ with $n \equiv 2, 3 \pmod 4$ and $n \ge 7$. By similar arguments as in Lemma 4.3, there is a pair of disjoint 4-GDDs of type $12^{n-2}24^1$ with $\sum_{i=1}^{x}\alpha_i$ common triangles, where $x = 3(n^2-n-2)/4$ and $\alpha_i \in \{0, 16\}$ for $1 \le i \le x$. By Construction 2.2, filling in the holes by a pair of $S(2, 4, 16)$s with $\beta_j + 4$ $(1 \le j \le n-2)$ common triangles and exactly one common block from Lemma 3.4, and a pair of disjoint $S(2, 4, 28)$s with β_{n-1} common triangles from Lemmas 3.1 and 3.7, we have a pair of disjoint $S(2, 4, 12n+4)$s with $\sum_{i=1}^{x}\alpha_i + \sum_{j=1}^{n-2}\beta_j + \beta_{n-1}$ common triangles, where $\beta_j \in J_1^*(16)$ for $1 \le j \le n-2$ and $\beta_{n-1} \in \{0, 1, 2, 3, 63\} \subseteq J_T^*(28)$. It is readily checked that for any integer $a \in [0, b_v]$, a can be written as the form of $\sum_{i=1}^{x}\alpha_i + \sum_{j=1}^{n-2}\beta_j + \beta_{n-1}$, where $\alpha_i \in \{0, 16\}$ $(1 \le i \le x)$, $\beta_j \in J_1^*(16) = [0, 19]$ $(1 \le j \le n-2)$, $\beta_{n-1} \in \{0, 1, 2, 3, 63\}$.

When $v = 76$, start from a 5-GDD of type 5^5 by Lemma 2.3. Give each point of the GDD weight 3. By Lemma 3.8, there is a pair of disjoint 4-GDDs of type 3^5 with α common triangles, $\alpha \in \{0, 15\}$. Then apply Construction 2.1 to obtain a pair of disjoint 4-GDDs of type 15^5 with $\sum_{i=1}^{25}\alpha_i$ common triangles, where $\alpha_i \in \{0, 15\}$ for $1 \le i \le 25$. By Construction 2.2, filling in the holes by a pair of disjoint $S(2, 4, 16)$s with β_j $(1 \le j \le 5)$ common triangles from Lemma 3.5, we have a pair of disjoint $S(2, 4, 76)$s with $\sum_{i=1}^{25}\alpha_i + \sum_{j=1}^{5}\beta_j$ common triangles, where $\beta_j \in J_T^*(16)$ for $1 \le j \le 5$. It is readily checked that for any integer $a \in [0, b_{76}]$, a can be written as the form of $\sum_{i=1}^{25}\alpha_i + \sum_{j=1}^{5}\beta_j$, where $\alpha_i \in \{0, 15\}$ $(1 \le i \le 25)$, $\beta_j \in J_T^*(16) = [0, 20]$ $(1 \le j \le 5)$. $\square$

Lemma 4.5. *Let $v \ge 52$ be a positive integer such that $v \equiv 1, 4 \pmod{12}$ and $v \not\equiv 4, 13 \pmod{36}$. Then $J_T^*(v) = [0, b_v]$.*

Proof. Let $v = 3n+1$ where $n \equiv 0, 1 \pmod 4$, $n \not\equiv 1, 4 \pmod{12}$ and $n \ge 17$. By Lemma 2.4 there exists a $(n, \{4, 5\}, 1)$-PBD, which must contain at least one block of size 5. Take a point from a block of size 5 and delete this point to obtain a $\{4, 5\}$-GDD of type 4^s3^t $(4s+3t = n-1, s \ge 1)$, which contains x blocks of size 5 and $y = [n(n-1)-20(x+s)]/12-t$ blocks of size 4. Give each point of the GDD weight 3. By Lemma 3.8 there is a pair of disjoint 4-GDDs of type 3^5 with 15 common triangles. By Lemma 3.9 there exists a pair of disjoint 4-GDDs of type 3^4 with 9 common triangles. Then apply Construction 2.1 to obtain a pair of disjoint 4-GDDs of type 12^s9^t with $15x+9y = 3n(n-1)/4-15s-9t$ common triangles. By Construction 2.2, filling in the holes by a pair of $S(2, 4, 13)$s with $\beta_i + 4$ $(1 \le i \le t)$ common triangles and exactly one common block from Lemma 3.2, a pair of $S(2, 4, 16)$s with $\beta_i + 4$ $(t+1 \le i \le s+t-1)$ common triangles and exactly one common block from Lemma 3.4, and a pair of disjoint $S(2, 4, 16)$s with β_{s+t} common triangles from Lemma 3.5, we then have a pair of disjoint $S(2, 4, v)$s (note that $12s+9t+4 = 3n+1 = v$) with $3n(n-1)/4-15s-9t+\sum_{i=1}^{t}\beta_i + \sum_{i=t+1}^{s+t-1}\beta_i + \beta_{s+t}$ common triangles, where $\beta_i \in J_1^*(13)$ for $1 \le i \le t$, $\beta_i \in J_1^*(16)$ for $t+1 \le i \le s+t-1$, and $\beta_{s+t} \in J_T^*(16)$. It is readily checked that for any integer $a \in [b_v - (8t+19(s-1)+20), b_v]$, a can be written as the form of $3n(n-1)/4-15s-9t+\sum_{i=1}^{t+s}\beta_i$, where $\beta_i \in [4, 12]$ for $1 \le i \le t$, $\beta_i \in [0, 19]$ for $t+1 \le i \le s+t-1$, and $\beta_{s+t} \in [0, 20]$. Hence $[b_v - (8t+19(s-1)+20), b_v] \in J_T^*(v)$. Note that $8t+19(s-1)+20 = [8(3t+4s+1)+25s-5]/3 \ge (8n+20)/3 = 8(v-1)/9+20/3 \ge (v-1)/12$. The conclusion follows by combining the results from Lemmas 4.1, 4.3 and 4.4. $\square$

Lemma 4.6. *If $s \in J_T^*(v)$, then $v(2v+1)/3+s \in J_T^*(3v+1)$. Furthermore, if $J_T^*(v) = [0, b_v]$ and $v \ge 16$, then $J_T^*(3v+1) = [0, b_{3v+1}]$.*

Proof. The assumption of $s \in J_T^*(v)$ implies that $v \equiv 1, 4 \pmod{12}$ and hence $2v+1 \equiv 3 \pmod 6$. There exists a KTS$(2v+1)$ $(X, \mathcal{A})$, where $\mathcal{A}$ can be partitioned into v parallel classes on X: $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_v$. By assumption there is a pair of disjoint $S(2, 4, v)$s $(Y, \mathcal{C}_1)$ and $(Y, \mathcal{C}_2)$ with s common triangles, where $Y = \{\infty_1, \infty_2, \ldots, \infty_v\}$. Define two block sets

$$\mathcal{B}_1 = \bigcup_{j=1}^{v}\{\{\infty_j\} \cup T : T \in \mathcal{P}_j\}$$

and

$$\mathcal{B}_2 = \bigcup_{j=1}^{v}\{\{\infty_{j+1}\} \cup T : T \in \mathcal{P}_j\},$$

where the subscript of ∞_{j+1} is calculated modulo v. Then $(X \cup Y, \mathcal{B}_1 \cup \mathcal{C}_1)$ and $(X \cup Y, \mathcal{B}_2 \cup \mathcal{C}_2)$ are $S(2, 4, 3v+1)$s. It is readily checked that they are disjoint and $|T(\mathcal{B}_1 \cup \mathcal{C}_1) \bigcap T(\mathcal{B}_2 \cup \mathcal{C}_2)| = |T(\mathcal{B}_1) \bigcap T(\mathcal{B}_2)| + |T(\mathcal{C}_1) \bigcap T(\mathcal{C}_2)| = v(2v+1)/3+s$. This completes the proof of the first assertion. Since $v(2v+1)/3 < b_{3v+1} - v/4$ and $b_{3v+1} = v(2v+1)/3 + b_v$, the last assertion follows by combining the results from Lemmas 4.1, 4.3 and 4.4. $\square$

Lemma 4.7. *There exists a pair of disjoint 4-GDDs of type 9^4 with i common triangles, where $i \in \{21, 57\}$.*

Proof. Let $I_9 = \{1, 2, \dots, 9\}$. Two disjoint $KTS(9)$s $(I_9, \cup_{j=1}^{4} \mathcal{P}_i(j))$, $i = 1, 2$, are constructed by listing their parallel classes $\mathcal{P}_i(j)$ $(1 \le i \le 2, 1 \le j \le 4)$ as follows:

123	147	159	168	124	135	169	178
456	258	267	249	389	268	237	259
789	369	348	357	567	479	458	346
$\mathcal{P}_1(1)$	$\mathcal{P}_1(2)$	$\mathcal{P}_1(3)$	$\mathcal{P}_1(4)$	$\mathcal{P}_2(1)$	$\mathcal{P}_2(2)$	$\mathcal{P}_2(3)$	$\mathcal{P}_2(4)$.

Two resolvable 3-GDDs of type 9^3 $(I_9 \times Z_3, \{I_9 \times \{k\} : k \in Z_3\}, \cup_{j=1}^{9} \mathcal{Q}_i(j))$, $i = 1, 2$, are shown by exhibiting their parallel classes $\mathcal{Q}_i(j)$ $(1 \le i \le 2, 1 \le j \le 9)$ as below: For $j = 1, 2, 3, 4$, define

$$\mathcal{Q}_i(2j-1) = \{\{(x, k), (y, k+1), (z, k+2)\} : \{x, y, z\} \in \mathcal{P}_i(j), x < y < z, k \in Z_3\},$$
$$\mathcal{Q}_i(2j) = \{\{(x, k+2), (y, k+1), (z, k)\} : \{x, y, z\} \in \mathcal{P}_i(j), x < y < z, k \in Z_3\},$$
$$\mathcal{Q}_1(9) = \mathcal{Q}_2(9) = \{\{(x, 0), (x, 1), (x, 2)\} : x \in I_9\}.$$

Let $X = (I_9 \times Z_3) \cup Y$ where $Y = \{\infty_1, \infty_1, \dots, \infty_9\}$. Let $\mathcal{G} = \{I_9 \times \{k\} : k \in Z_3\} \cup \{Y\}$. Take the permutations $\sigma = (1\,2\,3\,4\,5\,6\,7\,8\,9)$ and $\tau = (8\,9)$ on X. Define three block sets

$$\mathcal{A}_1 = \bigcup_{j=1}^{9} \{\{\infty_j\} \cup T : T \in \mathcal{Q}_1(j)\},$$
$$\mathcal{A}_2 = \bigcup_{j=1}^{9} \{\{\infty_{\sigma(j)}\} \cup T : T \in \mathcal{Q}_2(j)\},$$
$$\mathcal{A}_3 = \bigcup_{j=1}^{9} \{\{\infty_{\tau(j)}\} \cup T : T \in \mathcal{Q}_2(j)\}.$$

Then $(X, \mathcal{G}, \mathcal{A}_k)$ $(k = 1, 2, 3)$ is a 4-GDD of type 9^4. It is readily checked that $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$ and $|T(\mathcal{A}_1) \bigcap T(\mathcal{A}_2)| = 21$, $\mathcal{A}_1 \cap \mathcal{A}_3 = \emptyset$ and $|T(\mathcal{A}_1) \bigcap T(\mathcal{A}_3)| = 57$. $\square$

Lemma 4.8. $[0, 3] \cup [12, 129] \subseteq J_T^*(40)$.

Proof. By Lemmas 3.9 and 4.7 there exists a pair of disjoint 4-GDDs of type 9^4 with α common triangles, where $\alpha \in \{0, 21, 57, 81\}$. By Construction 2.2, filling in the holes by a pair of $S(2, 4, 13)$s with $\beta_i + 4$ $(1 \le i \le 3)$ common triangles and exactly one common block from Lemma 3.2, and a pair of disjoint $S(2, 4, 13)$s with β_4 common triangles from Lemma 3.3, we then have a pair of disjoint $S(2, 4, 40)$s with $\alpha + \sum_{i=1}^{4} \beta_i$ common triangles, where $\beta_i \in J_1^*(13)$ for $1 \le i \le 3$ and $\beta_4 \in J_T^*(13)$. It is readily checked that for any integer $a \in [12, 129]$, a can be written in the form of $\alpha + \sum_{i=1}^{4} \beta_i$, where $\alpha \in \{0, 21, 57, 81\}$, $\beta_i \in [4, 12]$ $(1 \le i \le 3)$ and $\beta_4 \in [0, 12] \setminus \{1, 2\}$. Combining the result from Lemma 3.1, we complete the proof. $\square$

Lemma 4.9. $[4, 11] \subseteq J_T^*(40)$.

Proof. Construct an $S(2, 4, 40)$ $(X, \mathcal{B})$ with $X = Z_{40}$. All blocks of $\mathcal{B}$ are divided into two parts. The first part consists of $\{i, 10+i, 20+i, 30+i\}$, $0 \le i \le 9$. Develop the following base blocks by $+1$ modulo 40 to obtain the second part of $\mathcal{B}$:

$$\{0, 1, 4, 13\}, \qquad \{0, 2, 7, 24\}, \qquad \{0, 6, 14, 25\}.$$

Consider the following permutations on X.

$$\begin{aligned}
\pi_4 = {} & (0\,29\,17\,34\,23\,35\,21\,24\,3\,14\,2\,22\,16\,4\,12)(1\,18\,26)(7\,30)(19\,20)\\
& (6\,31\,39\,28\,37\,9\,15\,8\,36\,33\,13\,32\,25\,38\,11\,10\,27),\\
\pi_5 = {} & (0\,37\,23\,34\,38\,10\,16\,12\,33\,26\,9\,24\,15)(1\,6\,7\,27\,17\,32\,28\,30\,36)(18\,21\,25)\\
& (2\,39\,20\,35\,31\,22\,13\,14\,19\,5\,8\,29\,3\,11),\\
\pi_6 = {} & (0\,20\,23\,3\,22\,33\,21\,15\,34\,26\,18\,6\,32\,30\,11\,4\,14\,7\,29\,8\,35\,2\,16\,13\,24\,27\,25\,19)\\
& (1\,38\,36\,39\,10\,37\,12\,17)(5\,31\,28),\\
\pi_7 = {} & (0\,15\,9\,35\,22\,1\,24\,20\,27\,37\,3\,33\,13\,26\,11\,19\,12\,16\,2\,28\,25\,23\,7\,38\,17\,30\,39\,8\,29\,18\,10)\\
& (4\,14\,21\,5\,32\,31)(34\,36),\\
\pi_8 = {} & (0\,13\,37\,23\,16\,1\,34\,8\,19\,21\,3\,5\,2\,11\,32\,30\,24\,4\,39\,10\,17\,22\,6)(12\,36\,14\,20)(18\,25)\\
& (9\,35\,31\,38\,27\,33\,26\,29),\\
\pi_9 = {} & (0\,13\,37\,30\,24\,4\,39\,10\,17\,22\,6)(1\,34\,8\,19\,21\,3\,5\,2\,11\,32\,14\,20\,12\,36\,23\,16)\\
& (9\,35\,31\,38\,27\,33\,26\,29)(18\,25),
\end{aligned}$$

Table 1
Permutations for $J_T^*(25)$ in Lemma 5.1.

$\pi_4 = (0\ 11\ 16\ 4\ 22\ 15\ 18\ 5\ 14\ 3\ 13\ 7\ 6\ 10\ 19\ 2\ 8\ 12\ 23\ 21\ 9\ 24\ 17)(1\ 20)$	
$\pi_5 = (0\ 11\ 16\ 4\ 22\ 15\ 18\ 5\ 14\ 3\ 13\ 7\ 6\ 10\ 19\ 2\ 8\ 12\ 23\ 1\ 20\ 9\ 24\ 21\ 17)$	
$\pi_6 = (0\ 11\ 16\ 4\ 22\ 2\ 8\ 12\ 23\ 17)(1\ 20)(3\ 13\ 7\ 6\ 10\ 19\ 9\ 24\ 21\ 15\ 18\ 5\ 14)$	
$\pi_7 = (0\ 11\ 16\ 4\ 22\ 2\ 8\ 12\ 23\ 21\ 17)(1\ 20)(3\ 13\ 7\ 6\ 10\ 19\ 9\ 24\ 15\ 18\ 5\ 14)$	
$\pi_8 = (0\ 11\ 16\ 4\ 22\ 21\ 17)(1\ 20)(2\ 8\ 12\ 23)(3\ 13\ 7\ 6\ 10\ 19\ 9\ 24\ 15\ 18\ 5\ 14)$	
$\pi_9 = (0\ 11\ 16\ 4\ 22\ 17)(1\ 20\ 2\ 8\ 12\ 23\ 21)(3\ 13\ 7\ 6\ 10\ 19\ 9\ 24\ 15\ 18\ 5\ 14)$	
$\pi_{10} = (0\ 11\ 16\ 4\ 22\ 1\ 20\ 2\ 8\ 12\ 23\ 21\ 15\ 18\ 5\ 14\ 3\ 13\ 7\ 6\ 10\ 19\ 9\ 24\ 17)$	
$\pi_{11} = (0\ 11\ 16\ 4\ 22\ 15\ 18\ 5\ 14\ 3\ 13\ 7\ 6\ 10\ 19\ 9\ 24\ 1\ 20\ 2\ 8\ 12\ 23\ 17)$	
$\pi_{12} = (0\ 11\ 16\ 4\ 22\ 21\ 2\ 8\ 12\ 23\ 1\ 20\ 15\ 18\ 5\ 14\ 3\ 13\ 7\ 6\ 10\ 19\ 9\ 24\ 17)$	
$\pi_{13} = (0\ 11\ 16\ 4\ 22\ 1\ 20\ 15\ 18\ 5\ 14\ 3\ 13\ 7\ 6\ 10\ 19\ 9\ 24\ 21\ 17)(2\ 8\ 12\ 23)$	
$\pi_{14} = (0\ 11\ 16\ 4\ 22\ 21\ 17)(1\ 20\ 15\ 18\ 5\ 14\ 3\ 13\ 7\ 6\ 10\ 19\ 9\ 24)(2\ 8\ 12\ 23)$	
$\pi_{15} = (0\ 11\ 16\ 4\ 22\ 1\ 20\ 15\ 18\ 5\ 14\ 3\ 13\ 7\ 6\ 10\ 19\ 9\ 24\ 17)(2\ 8\ 12\ 23)$	
$\pi_{16} = (0\ 11\ 16\ 4\ 22\ 17)(1\ 20\ 15\ 18\ 5\ 14\ 3\ 13\ 7\ 6\ 10\ 19\ 9\ 24)(2\ 8\ 12\ 23)$	
$\pi_{17} = (16\ 18\ 22\ 20\ 17\ 19\ 23\ 21\ 24)$	$\pi_{18} = (16\ 18\ 22\ 21\ 24)(17\ 19\ 23\ 20)$
$\pi_{19} = (16\ 18\ 22\ 20\ 21\ 17\ 19\ 23\ 24)$	$\pi_{20} = (16\ 18\ 22\ 21\ 17\ 19\ 23\ 20\ 24)$
$\pi_{21} = (16\ 18\ 22\ 20)(17\ 19\ 24\ 23\ 21)$	$\pi_{22} = (16\ 18\ 22\ 20)(17\ 19\ 24\ 21)$
$\pi_{23} = (16\ 18\ 22\ 17\ 19\ 24\ 23\ 21\ 20)$	$\pi_{24} = (16\ 18\ 22\ 17\ 19\ 24\ 21\ 20)$
$\pi_{25} = (16\ 18\ 22\ 21)(17\ 19\ 24\ 23\ 20)$	$\pi_{26} = (16\ 18\ 22\ 21\ 20\ 17\ 19\ 24\ 23)$
$\pi_{27} = (16\ 18\ 22\ 17\ 19\ 24\ 23\ 21)$	$\pi_{28} = (16\ 18\ 22\ 21)(17\ 19\ 24)$
$\pi_{29} = (16\ 18\ 22\ 17\ 19\ 24\ 23\ 20\ 21)$	$\pi_{30} = (19\ 23\ 20\ 21\ 22)$
$\pi_{31} = (19\ 23\ 24\ 20\ 21\ 22)$	$\pi_{32} = (19\ 23)(20\ 21\ 22)$
$\pi_{33} = (19\ 23\ 24)(20\ 21\ 22)$	$\pi_{34} = (19\ 23\ 22)(20\ 21\ 24)$
$\pi_{35} = (20\ 21\ 22\ 23\ 24)$	$\pi_{36} = (20\ 21\ 22\ 24\ 23)$
$\pi_{37} = (20\ 21\ 23\ 24\ 22)$	$\pi_{38} = (20\ 21\ 23)$
$\pi_{39} = (20\ 21\ 24)$	$\pi_{40} = (20\ 24)(22\ 23)$
$\pi_{41} = (19\ 21)(23\ 24)$	$\pi_{42} = (19\ 23)$
$\pi_{43} = (19\ 24)$	$\pi_{50} = (1)$

$$\begin{aligned}\pi_{10} &= (2\ 11\ 32\ 14\ 20\ 12\ 36\ 26\ 29\ 9\ 35\ 38\ 27\ 33\ 8\ 19\ 21\ 3\ 5)(1\ 34\ 23\ 16)(18\ 25)\\ &\quad (0\ 13\ 37\ 30\ 24\ 4\ 39\ 10\ 17\ 22\ 6),\\ \pi_{11} &= (0\ 12\ 32\ 16\ 26\ 6\ 36\ 21\ 19\ 17\ 37\ 13\ 38\ 14\ 23\ 34\ 39\ 35\ 2\ 9\ 15\ 33\ 25\ 7\ 31\ 3\ 24\ 30\ 10\ 18\ 20\ 8\ 29\ 4)\\ &\quad (5\ 27\ 11\ 28),\end{aligned}$$

It is readily checked that $(X, \pi_k\mathcal{B})$ and $(X, \mathcal{B})$ are disjoint and $|T(\pi_k\mathcal{B}) \bigcap T(\mathcal{B})| = k$ for each $k \in [4, 11]$. $\square$

Lemma 4.10. $b_{40} \in J_T^*(40)$.

Proof. Two $S(2, 4, 40)$s $(Z_{39} \cup \{\infty\}, \mathcal{B}_i)$, $i = 1, 2$, are constructed by listing their base blocks as below. All other blocks are obtained by developing these bases by $+3$ modulo 39, where $\infty + 3 = \infty$.

$$\begin{array}{lllll}\mathcal{B}_1: & \{0, 14, 20, 29\}, & \{0, 16, 23, 35\}, & \{0, 1, 2, 3\}, & \{0, 4, 7, 31\},\\ & \{0, 5, 18, 27\}, & \{0, 6, 19, 28\}, & \{0, 8, 10, 15\}, & \{1, 5, 23, 34\},\\ & \{1, 14, 17, 22\}, & \{\infty, 0, 11, 25\}. & & \\ \mathcal{B}_2: & \{0, 1, 20, 29\}, & \{1, 2, 18, 34\}, & \{0, 2, 3, 7\}, & \{1, 3, 13, 16\},\\ & \{0, 5, 18, 30\}, & \{0, 6, 25, 34\}, & \{0, 8, 11, 15\}, & \{1, 5, 11, 23\},\\ & \{1, 14, 22, 38\}, & \{\infty, 11, 25, 33\}. & & \end{array}$$

It is readily checked $\mathcal{B}_1 \bigcap \mathcal{B}_2 = \emptyset$ and $|T(\mathcal{B}_1) \bigcap T(\mathcal{B}_2)| = 130$. $\square$

Lemma 4.11. $J_T^*(85) = [0, b_{85}]$.

Proof. Start from an $S(2, 4, 28)$ and delete a point from this design to obtain a 4-GDD of type 3^9, which contains 54 blocks of size 4. Give each point of the GDD weight 3. By Lemma 3.9, there is a pair of disjoint 4-GDDs of type 3^4 with 9 common triangles. Then apply Construction 2.1 to obtain a pair of disjoint 4-GDDs of type 9^9 with 486 common triangles. By Construction 2.2, filling in the holes by a pair of $S(2, 4, 13)$s with $\beta_i + 4$ $(1 \le i \le 8)$ common triangles and exactly one common block from Lemma 3.2, and a pair of disjoint $S(2, 4, 13)$s with β_9 common triangles from Lemma 3.3, we then have a pair of disjoint $S(2, 4, 85)$s with $486 + \sum_{i=1}^{9} \beta_i$ common triangles, where $\beta_i \in J_1^*(13)$ for $1 \le i \le 8$ and $\beta_9 \in J_T^*(13)$. It is readily checked that for any integer $a \in [518, 594]$, a can be written in the form of $486 + \sum_{i=1}^{9} \beta_i$, where $\beta_i \in [4, 12]$ for $1 \le i \le 8$ and $\beta_9 \in [0, 12] \setminus \{1, 2\}$. Combining the result from Lemma 4.3 we have $[0, b_{85} - 1] \subseteq J_T^*(85)$.

Since $b_{28} \in J_T^*(28)$ from Lemma 3.7, by Lemma 4.6 with $v = 28$, we have $532 + b_{28} = b_{85} \in J_T^*(85)$. $\square$

5. The cases of $v = 25, 28, 37$

Lemma 5.1. (1) $[4, 43] \cup \{50\} \subseteq J_T^*(25)$.
(2) $[4, 56] \cup \{63\} \subseteq J_T^*(28)$.
(3) $[4, 106] \cup \{111\} \subseteq J_T^*(37)$.

Table 2
Permutations for $J_T^*(28)$ in Lemma 5.1.

$\pi_4 = (0\ 26\ 6\ 22\ 13\ 11\ 2\ 12\ 17\ 4\ 18\ 20\ 10\ 24\ 3\ 27\ 1)(5\ 23)(7\ 15)(8\ 21\ 9)(14\ 16\ 19)$	
$\pi_5 = (0\ 26\ 6\ 22\ 13\ 11\ 2\ 12\ 17\ 4\ 18\ 20\ 10\ 24\ 3\ 27\ 1)(5\ 23\ 25)(7\ 15)(8\ 21\ 9)(14\ 16\ 19)$	
$\pi_6 = (0\ 26\ 1)(2\ 12\ 17\ 4\ 18\ 20\ 10\ 24\ 3\ 27\ 6\ 22\ 9\ 8\ 21\ 13\ 11)(5\ 23)(7\ 15)(14\ 16\ 19)$	
$\pi_7 = (0\ 26\ 6\ 22\ 9\ 8\ 21\ 13\ 11\ 2\ 12\ 17\ 4\ 18\ 20\ 10\ 24\ 3\ 27\ 5\ 23\ 25\ 1)(7\ 15)(14\ 16\ 19)$	
$\pi_8 = (0\ 12\ 14\ 18\ 22\ 7\ 23\ 10\ 19\ 17\ 3\ 26\ 25\ 8\ 4)(2\ 9\ 15\ 16\ 20\ 11\ 5\ 24\ 27\ 6\ 21)$	
$\pi_9 = (0\ 12\ 14\ 18\ 22\ 7\ 23\ 27\ 6\ 21\ 2\ 9\ 15\ 16\ 20\ 11\ 5\ 24\ 25\ 8\ 4)(3\ 26\ 10\ 19\ 17)$	
$\pi_{10} = (0\ 12\ 14\ 18\ 22\ 10\ 19\ 17\ 3\ 26\ 25\ 8\ 4)(2\ 9\ 15\ 16\ 20\ 11\ 5\ 24\ 7\ 23\ 27\ 6\ 21)$	
$\pi_{11} = (0\ 12\ 14\ 18\ 22\ 10\ 19\ 17\ 3\ 26\ 6\ 21\ 2\ 9\ 15\ 16\ 20\ 11\ 5\ 24\ 7\ 23\ 27\ 8\ 4)$	
$\pi_{12} = (0\ 12\ 14\ 18\ 22\ 10\ 19\ 17\ 3\ 26\ 25\ 7\ 23\ 27\ 6\ 21\ 2\ 9\ 15\ 16\ 20\ 11\ 5\ 24\ 8\ 4)$	
$\pi_{13} = (0\ 12\ 14\ 18\ 22\ 10\ 19\ 17\ 3\ 26\ 6\ 21\ 2\ 9\ 15\ 16\ 20\ 11\ 5\ 24\ 25\ 7\ 23\ 27\ 8\ 4)$	
$\pi_{14} = (0\ 12\ 14\ 18\ 22\ 10\ 19\ 17\ 3\ 26\ 8\ 4)(2\ 9\ 15\ 16\ 20\ 11\ 5\ 24\ 25\ 7\ 23\ 27\ 6\ 21)$	
$\pi_{15} = (17\ 18\ 20\ 21\ 19\ 24\ 23\ 25\ 26\ 22\ 27)$	$\pi_{16} = (17\ 18\ 20\ 21\ 27\ 25\ 26)(19\ 24\ 23\ 22)$
$\pi_{17} = (17\ 18\ 20\ 21\ 27\ 22\ 23\ 19\ 24\ 25\ 26)$	$\pi_{18} = (17\ 18\ 20\ 21\ 27\ 25\ 26)(19\ 24)(22\ 23)$
$\pi_{19} = (19\ 21\ 27\ 25\ 26\ 23\ 20\ 22\ 24)$	$\pi_{20} = (19\ 21\ 27\ 23\ 20\ 22\ 24)(25\ 26)$
$\pi_{21} = (19\ 21\ 27)(20\ 22\ 24\ 25\ 26\ 23)$	$\pi_{22} = (19\ 21\ 27\ 23\ 25\ 26\ 20\ 22\ 24)$
$\pi_{23} = (19\ 21\ 27)(20\ 22\ 24)(23\ 25\ 26)$	$\pi_{24} = (19\ 21\ 27\ 20\ 22\ 24)(23\ 26\ 25)$
$\pi_{25} = (19\ 21\ 27\ 23\ 26\ 25)(20\ 22\ 24)$	$\pi_{26} = (19\ 21\ 27\ 23\ 26\ 20\ 22\ 24\ 25)$
$\pi_{27} = (19\ 21\ 27)(20\ 22\ 24\ 25)(23\ 26)$	$\pi_{28} = (19\ 21\ 27\ 23)(20\ 22\ 25\ 26)$
$\pi_{29} = (19\ 21\ 27\ 24\ 26\ 20\ 22\ 25\ 23)$	$\pi_{30} = (19\ 21\ 27)(20\ 22\ 25\ 24\ 23)$
$\pi_{31} = (19\ 21\ 27\ 24\ 26\ 20\ 22\ 25)$	$\pi_{32} = (19\ 21\ 27\ 23\ 24)(20\ 22\ 25)$
$\pi_{33} = (19\ 21\ 27\ 20\ 22\ 25\ 23\ 24)$	$\pi_{34} = (19\ 21\ 27\ 23\ 24\ 20\ 22\ 25)$
$\pi_{35} = (19\ 21\ 27)(20\ 22\ 25\ 23\ 24)$	$\pi_{36} = (19\ 21\ 27)(20\ 22\ 26)$
$\pi_{37} = (19\ 21)(20\ 23\ 27\ 24)$	$\pi_{38} = (19\ 21\ 24\ 20\ 23\ 27\ 22\ 25)$
$\pi_{39} = (19\ 21)(20\ 24)(23\ 26)$	$\pi_{40} = (19\ 21)(20\ 24)(22\ 25)$
$\pi_{41} = (19\ 21)(20\ 25\ 24\ 22)$	$\pi_{42} = (19\ 21)(20\ 25\ 22)$
$\pi_{43} = (19\ 21)(20\ 25\ 24)$	$\pi_{44} = (19\ 21\ 20\ 25\ 24)$
$\pi_{45} = (19\ 21)(20\ 27)$	$\pi_{46} = (19\ 21\ 24\ 20\ 27)$
$\pi_{47} = (19\ 22\ 25\ 21\ 20)$	$\pi_{48} = (19\ 22\ 20)(26\ 27)$
$\pi_{49} = (19\ 22\ 20)(24\ 25)$	$\pi_{50} = (19\ 22\ 27)$
$\pi_{51} = (19\ 23)$	$\pi_{52} = (23\ 25)$
$\pi_{53} = (18\ 20)$	$\pi_{54} = (24\ 26)$
$\pi_{55} = (23\ 24)$	$\pi_{56} = (21\ 24)$
$\pi_{63} = (1)$	

Proof. (1) Take a pair of $S(2, 4, 25)$s $(X, \mathcal{B}_i)$ $(i = 1, 2)$ listed in Lemma 3.6. Using the necessary permutation π_k for each $k \in [4, 43] \cup \{50\}$ given in Table 1, we can obtain disjoint designs $(X, \pi_k\mathcal{B}_2)$ and $(X, \mathcal{B}_1)$ where $|\pi_k T(\mathcal{B}_2) \bigcap T(\mathcal{B}_1)| = k$. For convenience we denote any element $(a, b) \in X$ by an integer $5a + b$.

(2) Take a pair of $S(2, 4, 28)$s $(X, \mathcal{B}_i)$ $(i = 1, 2)$ listed in Lemma 3.7. Using the necessary permutation π_k for each $k \in [4, 56] \cup \{63\}$ given in Table 2, we can obtain disjoint designs $(X, \pi_k\mathcal{B}_2)$ and $(X, \mathcal{B}_1)$ where $|\pi_k T(\mathcal{B}_2) \bigcap T(\mathcal{B}_1)| = k$.

(3) Construct a pair of $S(2, 4, 37)$s $(X, \mathcal{B}_i)$ $(i = 1, 2)$ on $X = Z_{37}$. Only base blocks are listed below. Develop these base blocks by $+1$ modulo 37 to obtain all blocks of $\mathcal{B}_i$, $i = 1, 2$.

$$\begin{array}{llll} \mathcal{B}_1: & \{0, 1, 3, 24\}, & \{0, 4, 9, 15\}, & \{0, 7, 17, 25\}. \\ \mathcal{B}_2: & \{0, 1, 14, 17\}, & \{0, 2, 6, 11\}, & \{0, 7, 15, 25\}. \end{array}$$

Using necessary permutation π_k for each $k \in [4, 99] \cup \{111\}$ on X given in Table 3, we can obtain disjoint designs $(X, \pi_k\mathcal{B}_2)$ and $(X, \mathcal{B}_1)$ where $|\pi_k T(\mathcal{B}_2) \bigcap T(\mathcal{B}_1)| = k$.

Construct another $S(2, 4, 37)$ $(X, \mathcal{B})$ with $X = Z_{37}$ as follows. This construction can be found in [12].

{0, 1, 3, 9},	{0, 2, 8, 12},	{0, 4, 5, 7},	{0, 6, 10, 11},	{0, 13, 18, 30},	{0, 14, 17, 34},	{0, 15, 26, 31},
{0, 16, 27, 32},	{0, 19, 22, 35},	{0, 20, 23, 36},	{0, 21, 24, 33},	{0, 25, 28, 29},	{1, 2, 4, 10},	{1, 5, 6, 8},
{1, 7, 11, 12},	{1, 13, 21, 32},	{1, 14, 22, 33},	{1, 15, 23, 34},	{1, 16, 24, 35},	{1, 17, 25, 36},	{1, 18, 26, 29},
{1, 19, 27, 30},	{1, 20, 28, 31},	{2, 3, 30, 34},	{2, 5, 23, 27},	{2, 6, 7, 9},	{2, 11, 21, 25},	{2, 13, 16, 33},
{2, 14, 19, 31},	{2, 15, 18, 35},	{2, 17, 20, 29},	{2, 22, 24, 32},	{2, 26, 28, 36},	{3, 4, 32, 36},	{3, 5, 24, 28},
{3, 6, 15, 19},	{3, 7, 8, 10},	{3, 11, 22, 26},	{3, 12, 14, 18},	{3, 13, 20, 35},	{3, 16, 17, 31},	{3, 21, 27, 29},
{3, 23, 25, 33},	{4, 6, 29, 33},	{4, 8, 9, 11},	{4, 12, 31, 35},	{4, 13, 15, 28},	{4, 14, 20, 27},	{4, 16, 18, 23},
{4, 17, 19, 24},	{4, 21, 26, 30},	{4, 22, 25, 34},	{5, 9, 10, 12},	{5, 11, 13, 17},	{5, 14, 15, 29},	{5, 16, 25, 26},
{5, 18, 19, 33},	{5, 20, 21, 22},	{5, 30, 31, 36},	{5, 32, 34, 35},	{6, 12, 16, 20},	{6, 13, 14, 36},	{6, 17, 18, 32},
{6, 21, 23, 31},	{6, 22, 28, 30},	{6, 24, 26, 34},	{6, 25, 27, 35},	{7, 13, 25, 31},	{7, 14, 26, 32},	{7, 15, 27, 33},
{7, 16, 28, 34},	{7, 17, 21, 35},	{7, 18, 22, 36},	{7, 19, 23, 29},	{7, 20, 24, 30},	{8, 13, 24, 29},	{8, 14, 25, 30},
{8, 15, 20, 32},	{8, 16, 19, 36},	{8, 17, 28, 33},	{8, 18, 21, 34},	{8, 22, 27, 31},	{8, 23, 26, 35},	{9, 13, 27, 34},
{9, 14, 28, 35},	{9, 15, 21, 36},	{9, 16, 22, 29},	{9, 17, 23, 30},	{9, 18, 24, 31},	{9, 19, 25, 32},	{9, 20, 26, 33},
{10, 13, 19, 26},	{10, 14, 16, 21},	{10, 15, 17, 22},	{10, 18, 20, 25},	{10, 23, 28, 32},	{10, 24, 27, 36},	{10, 29, 30, 35},
{10, 31, 33, 34},	{11, 14, 23, 24},	{11, 15, 16, 30},	{11, 18, 27, 28},	{11, 19, 20, 34},	{11, 29, 31, 32},	{11, 33, 35, 36},
{12, 13, 22, 23},	{12, 15, 24, 25},	{12, 17, 26, 27},	{12, 19, 21, 28},	{12, 29, 34, 36},	{12, 30, 32, 33}.	

Using the necessary permutation π_k for each $k \in [100, 106]$ on X given in Table 3, we can obtain disjoint designs $(X, \pi_k\mathcal{B})$ and $(X, \mathcal{B})$ where $|\pi_k T(\mathcal{B}) \bigcap T(\mathcal{B})| = k$. $\square$

Table 3
Permutations for $J_T^*(37)$ in Lemma 5.1.

$\pi_4 = (0\ 22\ 13)(1\ 9\ 14\ 35\ 27\ 29\ 3\ 34\ 32\ 12\ 15\ 4\ 19\ 36\ 24\ 17\ 5)(2\ 16\ 23\ 10\ 8\ 6\ 31\ 33)(7\ 25\ 30)(11\ 28\ 21\ 20)$

$\pi_5 = (0\ 22\ 13)(1\ 9\ 14\ 35\ 3\ 34\ 32\ 27\ 29\ 33\ 2\ 16\ 23\ 10\ 8\ 6\ 31\ 12\ 15\ 4\ 19\ 36\ 24\ 17\ 5)(7\ 25\ 30)(11\ 28\ 21\ 20)$

$\pi_6 = (0\ 4\ 23\ 10\ 15\ 30\ 33\ 6\ 1\ 36\ 2\ 14\ 17\ 13\ 35\ 18\ 19\ 27\ 9\ 3\ 31\ 11\ 8\ 5\ 28\ 20\ 25\ 29\ 26\ 7\ 22\ 32\ 34)(12\ 21\ 24\ 16)$

$\pi_7 = (0\ 4\ 23\ 10\ 15\ 30\ 33\ 6\ 1\ 36\ 2\ 14\ 17\ 13\ 35\ 18\ 19\ 27\ 9\ 3\ 31\ 34)(5\ 28\ 20\ 25\ 29\ 26\ 7\ 22\ 32\ 11\ 8)(12\ 21\ 24\ 16)$

$\pi_8 = (0\ 18\ 11\ 6)(1\ 32\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(3\ 4\ 31\ 26\ 19\ 12\ 22\ 30\ 33\ 36\ 35\ 9\ 34\ 10)(5\ 16)$

$\pi_9 = (0\ 18\ 11\ 6)(1\ 32\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(3\ 4\ 31\ 35\ 36\ 26\ 19\ 12\ 22\ 30\ 33\ 9\ 34\ 10)(5\ 16)$

$\pi_{10} = (0\ 18\ 11\ 6)(1\ 32\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(3\ 4\ 31\ 35\ 10)(5\ 16)(9\ 34\ 26\ 19\ 12\ 22\ 30\ 33)$

$\pi_{11} = (0\ 18\ 11\ 6)(1\ 32\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(3\ 4\ 31\ 35\ 26\ 19\ 12\ 22\ 30\ 33\ 10)(5\ 16)(9\ 34)$

$\pi_{12} = (0\ 18\ 11\ 6)(1\ 32\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(3\ 4\ 31\ 35\ 9\ 34\ 26\ 19\ 12\ 22\ 30\ 33\ 10)(5\ 16)$

$\pi_{13} = (0\ 18\ 11\ 6)(1\ 32\ 9\ 34\ 10\ 3\ 4\ 31\ 35\ 26\ 19\ 12\ 22\ 30\ 33\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)$

$\pi_{14} = (0\ 18\ 11\ 6)(1\ 32\ 9\ 34\ 26\ 19\ 12\ 22\ 30\ 33\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(3\ 4\ 31\ 35\ 36\ 10)(5\ 16)$

$\pi_{15} = (0\ 18\ 11\ 6)(1\ 32\ 9\ 34\ 36\ 26\ 19\ 12\ 22\ 30\ 33\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(3\ 4\ 31\ 35\ 10)(5\ 16)$

$\pi_{16} = (0\ 18\ 11\ 6)(1\ 32\ 9\ 34\ 26\ 19\ 12\ 22\ 30\ 33\ 10\ 3\ 4\ 31\ 35\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)$

$\pi_{17} = (0\ 18\ 11\ 6)(1\ 32\ 10\ 3\ 4\ 31\ 35\ 26\ 19\ 12\ 22\ 30\ 33\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)(9\ 34\ 36)$

$\pi_{18} = (0\ 18\ 11\ 6)(1\ 32\ 26\ 19\ 12\ 22\ 30\ 33\ 10\ 3\ 4\ 31\ 35\ 36\ 9\ 34\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)$

$\pi_{19} = (0\ 18\ 11\ 6)(1\ 32\ 26\ 19\ 12\ 22\ 30\ 33\ 10\ 3\ 4\ 31\ 35\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)(9\ 34\ 36)$

$\pi_{20} = (0\ 18\ 11\ 6)(1\ 32\ 36\ 9\ 34\ 26\ 19\ 12\ 22\ 30\ 33\ 10\ 3\ 4\ 31\ 35\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)$

$\pi_{21} = (0\ 18\ 11\ 6)(1\ 32\ 9\ 34\ 35\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(3\ 4\ 31\ 36\ 10)(5\ 16)(12\ 22\ 30\ 33\ 26\ 19)$

$\pi_{22} = (0\ 18\ 11\ 6)(1\ 32\ 9\ 34\ 35\ 10\ 3\ 4\ 31\ 36\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)(12\ 22\ 30\ 33\ 26\ 19)$

$\pi_{23} = (0\ 18\ 11\ 6)(1\ 32\ 10\ 3\ 4\ 31\ 36\ 9\ 34\ 26\ 19\ 12\ 22\ 30\ 33\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)$

$\pi_{24} = (0\ 18\ 11\ 6)(1\ 32\ 35\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(3\ 4\ 31\ 36\ 9\ 34\ 26\ 19\ 12\ 22\ 30\ 33\ 10)(5\ 16)$

$\pi_{25} = (0\ 18\ 11\ 6)(1\ 32\ 9\ 34\ 26\ 19\ 12\ 22\ 30\ 35\ 33\ 10\ 3\ 4\ 31\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)$

$\pi_{26} = (0\ 18\ 11\ 6)(1\ 32\ 26\ 19\ 12\ 22\ 30\ 35\ 33\ 10\ 3\ 4\ 31\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)(9\ 34\ 36)$

$\pi_{27} = (0\ 18\ 11\ 6)(1\ 32\ 36\ 9\ 34\ 26\ 19\ 12\ 22\ 30\ 35\ 33\ 10\ 3\ 4\ 31\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)$

$\pi_{28} = (0\ 18\ 11\ 6)(1\ 32\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(3\ 4\ 31\ 36\ 9\ 34\ 26\ 19\ 12\ 22\ 30\ 35\ 33\ 10)(5\ 16)$

$\pi_{29} = (0\ 18\ 11\ 6)(1\ 32\ 9\ 34\ 26\ 19\ 12\ 22\ 30\ 35\ 33\ 10\ 3\ 4\ 31\ 36\ 7\ 8\ 24\ 15\ 29\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)$

$\pi_{30} = (0\ 18\ 11\ 6)(1\ 32\ 33\ 26\ 19\ 12\ 22\ 30\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(3\ 4\ 31\ 36\ 9\ 34\ 35\ 10)(5\ 16)(7\ 8\ 24\ 15\ 29)$

$\pi_{31} = (0\ 18\ 11\ 6)(1\ 32\ 33\ 36\ 35\ 10\ 3\ 4\ 31\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)(7\ 8\ 24\ 15\ 29)(9\ 34\ 26\ 19\ 12\ 22\ 30)$

$\pi_{32} = (0\ 18\ 11\ 6)(1\ 32\ 33\ 10\ 3\ 4\ 31\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)(7\ 8\ 24\ 15\ 29)(9\ 34\ 36)(12\ 22\ 30\ 26\ 19)$

$\pi_{33} = (0\ 18\ 11\ 6)(1\ 32\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(3\ 4\ 31\ 36\ 9\ 34\ 26\ 19\ 12\ 22\ 30\ 35\ 33\ 10)(5\ 16)(7\ 8\ 24\ 15\ 29)$

$\pi_{34} = (0\ 18\ 11\ 6)(1\ 32\ 33\ 10\ 3\ 4\ 31\ 2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)(7\ 8\ 24\ 15\ 29\ 26\ 19\ 12\ 22\ 30\ 36\ 9\ 34\ 35)$

$\pi_{35} = (0\ 18\ 11\ 6)(1\ 32\ 33\ 10\ 3\ 4\ 31\ 36)(2\ 27\ 14\ 25\ 13\ 23\ 20\ 21\ 28)(5\ 16)(7\ 8\ 24\ 15\ 29\ 26\ 19\ 12\ 22\ 30\ 9\ 34\ 35)$

$\pi_{36} = (0\ 12\ 33\ 16\ 27\ 11\ 34\ 14\ 23\ 32\ 4)(2\ 9\ 15\ 28\ 5\ 35\ 29\ 21\ 19\ 17\ 3\ 24\ 26\ 6\ 31\ 10\ 18\ 20\ 8\ 36\ 25\ 7\ 30)$

$\pi_{37} = (0\ 12\ 33\ 21\ 19\ 17\ 3\ 24\ 26\ 6\ 31\ 10\ 18\ 20\ 8\ 36\ 25\ 7\ 30\ 16\ 27\ 14\ 23\ 32\ 5\ 35\ 29\ 2\ 9\ 15\ 28\ 4)(11\ 34)$

$\pi_{38} = (26\ 27\ 30\ 34\ 29\ 28\ 32)(31\ 35\ 33\ 36)$	$\pi_{39} = (26\ 27\ 30\ 35\ 33\ 36\ 31\ 34\ 29\ 28\ 32)$
$\pi_{40} = (26\ 27)(28\ 29)(30\ 35\ 33\ 31)(32\ 36\ 34)$	$\pi_{41} = (26\ 27)(28\ 29)(30\ 35\ 34\ 33\ 32\ 36\ 31)$
$\pi_{42} = (26\ 27)(28\ 29)(30\ 35\ 33\ 34\ 32\ 36\ 31)$	$\pi_{43} = (27\ 30\ 34\ 28\ 32\ 35\ 31\ 29\ 33\ 36)$
$\pi_{44} = (27\ 28)(29\ 32\ 36\ 34\ 33\ 31\ 35\ 30)$	$\pi_{45} = (27\ 28)(29\ 32\ 36\ 35\ 30\ 34\ 33\ 31)$
$\pi_{46} = (27\ 28)(29\ 32\ 36\ 35\ 33\ 31\ 30\ 34)$	$\pi_{47} = (27\ 28)(29\ 32\ 36\ 35)(30\ 34\ 33\ 31)$
$\pi_{48} = (27\ 28)(29\ 32\ 31\ 35\ 30\ 34)(33\ 36)$	$\pi_{49} = (27\ 28)(29\ 32\ 31\ 35)(30\ 34)(33\ 36)$
$\pi_{50} = (27\ 28)(29\ 32\ 33\ 36\ 30\ 34)(31\ 35)$	$\pi_{51} = (27\ 28)(29\ 32\ 36\ 33\ 30\ 34)(31\ 35)$
$\pi_{52} = (27\ 28)(29\ 32)(30\ 34\ 33)(31\ 36\ 35)$	$\pi_{53} = (27\ 28)(29\ 32)(30\ 34\ 35)(31\ 36\ 33)$
$\pi_{54} = (27\ 28)(29\ 32\ 31\ 36)(30\ 34\ 33)$	$\pi_{55} = (27\ 28)(29\ 32\ 31\ 36\ 35)(30\ 34)$
$\pi_{56} = (27\ 28)(29\ 32\ 31\ 36\ 30\ 34\ 35)$	$\pi_{57} = (27\ 28)(29\ 32\ 33)(30\ 34\ 31\ 36)$
$\pi_{58} = (27\ 28)(29\ 32\ 33\ 30\ 34)(31\ 36)$	$\pi_{59} = (27\ 28)(29\ 32\ 30\ 35\ 36\ 34\ 31)$
$\pi_{60} = (27\ 28)(29\ 32\ 31)(30\ 35\ 34)$	$\pi_{61} = (27\ 28)(29\ 32\ 31)(30\ 35\ 36)$
$\pi_{62} = (27\ 28)(29\ 32)(30\ 35\ 36\ 34\ 31)$	$\pi_{63} = (27\ 28)(29\ 32)(30\ 35\ 36\ 31)$
$\pi_{64} = (27\ 28)(29\ 32)(30\ 35\ 34\ 33)$	$\pi_{65} = (27\ 28)(29\ 32)(30\ 35\ 36\ 33)$
$\pi_{66} = (27\ 28)(29\ 32)(30\ 35\ 36\ 34)$	$\pi_{67} = (27\ 28)(29\ 32)(30\ 36\ 34\ 31)$
$\pi_{68} = (27\ 28)(29\ 33\ 36\ 34\ 30)$	$\pi_{69} = (27\ 28)(29\ 33\ 30)(31\ 32)$
$\pi_{70} = (27\ 28)(29\ 33\ 32\ 31)$	$\pi_{71} = (27\ 28)(29\ 33\ 32)(35\ 36)$
$\pi_{72} = (27\ 28)(29\ 33)(34\ 35)$	$\pi_{73} = (27\ 28)(29\ 33)(32\ 35)$
$\pi_{74} = (27\ 28)(29\ 33\ 34\ 31\ 32)$	$\pi_{75} = (27\ 28)(29\ 33)(31\ 34)$
$\pi_{76} = (27\ 28)(29\ 34\ 35\ 32)$	$\pi_{77} = (27\ 28)(29\ 34\ 33)$
$\pi_{78} = (27\ 28)(29\ 34\ 35)$	$\pi_{79} = (27\ 28)(29\ 36\ 30)$
$\pi_{80} = (27\ 28)(29\ 36\ 33)$	$\pi_{81} = (27\ 28\ 29)(35\ 36)$
$\pi_{82} = (27\ 28\ 29\ 32\ 33)$	$\pi_{83} = (27\ 28\ 29\ 36)$
$\pi_{84} = (27\ 28\ 31\ 29)$	$\pi_{85} = (27\ 28\ 31\ 30)$
$\pi_{86} = (27\ 28\ 31\ 32)$	$\pi_{87} = (27\ 28\ 34\ 30)$
$\pi_{88} = (27\ 28\ 34\ 31)$	$\pi_{89} = (33\ 36\ 34)$
$\pi_{90} = (32\ 36\ 35)$	$\pi_{91} = (31\ 32\ 33)$
$\pi_{92} = (31\ 34\ 35)$	$\pi_{93} = (30\ 36\ 33)$
$\pi_{94} = (29\ 35\ 36)$	$\pi_{95} = (28\ 33)$
$\pi_{96} = (28\ 36)$	$\pi_{97} = (32\ 36)$
$\pi_{98} = (31\ 32)$	$\pi_{99} = (29\ 36)$
$\pi_{100} = (0\ 11\ 2\ 5\ 6\ 8\ 4\ 3\ 10\ 9\ 1\ 12\ 7)$	$\pi_{101} = (0\ 11\ 2\ 5\ 6\ 8)(1\ 12\ 4\ 3\ 10\ 9\ 7)$
$\pi_{102} = (0\ 11\ 2\ 5\ 7\ 1\ 12\ 6\ 4\ 3\ 10\ 9\ 8)$	$\pi_{103} = (0\ 11\ 2\ 5\ 7\ 8)(1\ 12\ 6\ 4\ 3\ 10\ 9)$
$\pi_{104} = (0\ 12\ 4\ 6\ 8\ 10\ 9\ 7\ 1)(2\ 3\ 5\ 11)$	$\pi_{105} = (0\ 12\ 6\ 4\ 7\ 10\ 8\ 9\ 1)(2\ 5\ 3\ 11)$
$\pi_{106} = (0\ 7\ 9\ 1\ 10\ 8)(2\ 11\ 5)(3\ 12\ 6\ 4)$	$\pi_{111} = (1)$

6. Conclusion

Proof of Theorem 1.2. (1) When $v = 16, 49$, we have $J_T^*(v) = [0, b_v]$ by Lemmas 3.5 and 4.6. When $v = 40$, we have that $J_T^*(40) = [0, b_{40}]$ by combining the results of Lemmas 4.8–4.10. When $v \equiv 1, 4 \pmod{12}$, $v \not\equiv 13 \pmod{36}$ and $v \geq 52$, the conclusion follows by Lemmas 4.4 and 4.5. We only need to deal with the cases of $v \equiv 13 \pmod{36}$ and $v \geq 52$. We prove the theorem by induction. When $v = 85$, the conclusion follows by Lemma 4.11. When $v \equiv 13 \pmod{36}$ and $v \geq 121$, then $(v-1)/3 \equiv 4 \pmod{12}$ and $(v-1)/3 \geq 40$. By the induction, we have $J_T^*((v-1)/3) = [0, b_{(v-1)/3}]$. Apply Lemma 4.6 we obtain $J_T^*(v) = [0, b_v]$. This completes the first assertion.

By Lemma 3.3, (2) of Theorem 1.2 follows immediately. Combining the results of Lemmas 3.1 and 5.1, (3)–(5) of Theorem 1.2 hold. This completes the proof. □

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